Calculus on Manifolds

(Section 1) Functions on Euclidean Space

(subsection) Norm and Inner Product

(Def) Euclidean n-space \mathbf{R}^{n} is defined as the set of all n-tuples (x^{1}, \cdots, x^{n}) of real numbers x^{i} . An element of \mathbf{R}^{n} is often called a point in \mathbf{R}^{n} , and \mathbf{R}^{1}, \mathbf{R}^{2}, \mathbf{R}^{3} are often called the line, the plane, and space, respectively. If x denotes an element of \mathbf{R}^{n}, then x is an n-tuple of numbers, the ith one of which is denoted x^{i} ;. thus we can write x = (x^{1}, \cdots, x^{n}).

(Def) a point in \mathbf{R}^{n} is also called a vector in \mathbf{R}^{n}, because \mathbf{R}^{n}, with x + y = (x^{1} + y^{1}, \cdots, x^{n} + y^{n}) and ax = (ax^{1}, \cdots, ax^{n}) , as operations, is a vector space.

(Def) The length of a vector x, the norm |x| of x and defined by \x| = \sqrt{(x^{1})^{2} + \cdots + (x^{n})^{2}} .

(Thm 1.1) If x , y \in \mathbf{R}^{n} and a \in \mathbf{R} , then

|x| \ge 0, and |x| = 0 iff x = 0

| \sum\_{i = 1}^{n} x^{i} y^{i} | \le |x| \cdot |y| ; equality holds iff x and y are linearly depencent.

|x + y| \le |x| + |y| .

|ax| = |a| \cdot |x|.

(Def) The quantity \sum\_{i = 1}^{n} x^{i} y^{i} is called the inner product of x and y and denoted \langle x,y \rangle.

(Thm 1.2) If x, x\_{1}, x\_{2} and y, y\_{1}, y\_{2} are vectors in \mathbf{R}^{n} and a \in \mathbf{R}, then

\langle x , y \rangle = \langle y, x \rangle (symmetry)

\langle ax , y \rangle = \langle x , ay \rangle = a \langle x , y \rangle \langle x\_{1} + x\_{2} , y \rangle = \langle x\_{1} , y \rangle + \langle x\_{2} , y \rangle , \langle x , y\_{1} + y\_{2} \rangle = \langle x , y\_{1} \rangle + \langle x , y\_{2} \rangle (bilinearity)

\langle x , x \rangle \ge 0 and \langle x , x \rangle = 0 iff x = 0 ( positive definiteness)

|x| - \sqrt{ \langle x , x \rangle }

\langle x , y \rangle = \frac{|x + y|^{2} - |x – y|^{2}}{4} (polarization identity)

(Def) The vector (0, \cdots, 0) will be denoted simply 0. The usual basis of \mathbf{R}^{n} is e\_{1} , \cdots, e\_{n} where e\_{i} = (0, \cdots, 1, \cdots, 0) with the 1 in the ith place.

(Def) If T : \mathbf{R}^{n} \to \mathbf{R}^{m} is a linear transformation, the matrix of T with respect to the usual bases of \mathbf{R}^{n} and \mathbf{R}^{m} is the m \times n matrix A = (a\_{ij}) , where T(e\_{i}) = \sum\_{j = 1}^{m} a\_{ji}e\_{j} – the coefficients of T(e\_{i}) appear in the ith column of the matrix.

If S : \mathbf{R}^{m} \to \mathbf{R}^{p} has the p \times m matrix B, then S \bullet T has the p \times n matrix BA.

(Def) if x \in \mathbf{R}^{n} and y \in \mathbf{R}^{m}, then (x,y) denotes (x^{1} , \cdots, x^{n} , y^{1}, \cdots, y^{m}) \in \mathbf{R}^{n+m}

(subsection) Subsets of Euclidean Space

(Def) If A \subset \mathbf{R}^{m} and B \subset \mathbf{R}^{n}, then A \times B \subset \mathbf{R}^{m+n} is defined as the set of all (x,y) \in \mathbb{R}^{m+n} with x \in A and y \in B.

If A \subset \mathbf{R}^{m}, B \subset \mathbf{R}^{n} and C \subset \mathbf{R}^{p}, then (A \times B) \times C = A \times (B \times C) , and denoted simply A \times B \times C.

(Def) The set [a\_{1}, b\_{1}] \times \cdots \times [a\_{n} , b\_{n}] \subset \mathbf{R}^{n} is called a closed rectangle in \mathbf{R}^{n}, while the set (a\_{1}, b\_{1}) \times \cdots \times (a\_{n} , b\_{n}) \subset \mathbf{R}^{n} is called an open rectangle.

a set U \subset \mathbf{R}^{n} is called open if for each x \in U there is an open rectangle A s.t. x \in A \subset U. a subset C of \mathbf{R}^{n} is closed if \mathbf{R}^{n} – C is open.

(prop) If A \subset \mathbf{R}^{n} and x \in \mathbf{R}^{n}, then one of three possibilities must hold.

There is an open rectangle B s.t. x \in B \subset A. (Interior of A)

There is an open rectangle B s.t. x \in B \subset \mathbf{R}^{n} – A. (Exterior of A)

If B is any open rectangle with x \in B, then B contains points of both A and \mathbf{R}^{n} – A. (Boundary of A)

(Def) A collection \mathcal{O} of open sets is an open cover of A, or briefly , covers A , if every points x \inA is in some open set in the collection \mathcal{O}.

(Def) A set A is called compact if every open cover \mathcal{O} contains a finite subcollection of open sets which also covers A.

(Heine-Borel)(Thm 1.3) The closed interval [a,b] is compact.

(Thm 1.4) If B is compact and \mathcal{O} is an open cover of {x} \times B, then there is an open set U \subset \mathbf{R}^{n} containing x s.t. U \times B is covered by a finite number of sets in \mathcal{O}.

(Cor 1.5) If A \subset \mathbf{R}^{n} and B \subset \mathbf{R}^{m} are compact, then A \times B \subset \mathbf{R}^{n+m} is compact.

(Cor 1.6) A\_{1} \times \cdots \times A\_{k} is compact if each A\_{i} is. In particular, a closed rectangle in \mathbf{R}^{k} is compact.

(Cor 1.7) A closed bounded subset of \mathbf{R}^{n} is compact. The converse is also true.

(subsection) Functions and Continuity

(Def) A function from \mathbf{R}^{n} to \mathbf{R}^{m} (a vector-valued function of n variables) is a rule which associates to each point in \mathbf{R}^{n} some point in \mathbf{R}^{m}; the point a function f associates to x is denoted f(x).

Notation f : A \to \mathbf{R}^{m} indicates that f(x) is defined only for x in the set A, which is called the domain of f.

If B \subset A, we define f(B) as the set of all f(x) for x \in B, and if C \subset \mathbf{R}^{m} we define f^{-1}(C) = {x \in A : f(x) \in C} .

(Def) If f : A \to \mathbf{R}^{m} and g : B \to \mathbf{R}^{p}, where B \subset \mathbf{R}^{m}, then the composition g \bullet f is defined by g \bullet f (x) = g(f(x)) ; the domain of g \bullet f is A \cap f^{-1}(B). If f : A \to \mathbf{R}^{m} is 1-1, we define f^{-1} : f(A) \to \mathbf{R}^{n} by the requirement that the f^{-1} (z) is the unique x \in A with f(x) = z.

(Def) A function f : A \to \mathbf{R}^{m} determines m component functions f^{1}, \cdots, f^{m} : A \to \mathbf{R} by f(x) = (f^{1}(x), \cdots, f^{m}(x)) .

(Def) If \pi : \mathbf{R}^{n} \to \mathbf{R}^{n} is the identity function, \pi(x) = x, then \pi^{i} (x) = x^{i} ; the function \pi^{i} is called the ith projection function.

(Def \lim\_{x \to a} f(x) = b means, for every number \epsilon >0 there is a number \delta>0 s.t. |f(x) – b| < \epsilon for all x in the domain of f which satisfy 0< |x-a| < \delta.

(Def) A function f : A \to \mathbf{R}^{m} is called continuous at a \in A if \lim\_{x \to a} f(x) = f(a), and f is simply called continuous if it is continuous at each a \in A.

(Thm 1.8) If A \subset \mathbf{R}^{n}, a function f : A \to \mathbf{R}^{m} is continuous iff for every open set U \subset \mathbf{R}^{m} there is some open set V \subset \mathbf{R}^{n} s.t. f^{-1}(U) = V \cap A.

(Thm 1.9) If f : A \to \mathbf{R}^{m} is continuous, where A \subset \mathbf{R}^{n}, and A is compact, then f(A) \subset \mathbf{R}^{m} is compact.

(Def) If f : A \to \mathbf{R} is bounded, the extent to which f fails to be continuous at a \n A can be measured in a precise way. For \delta >0 let

M(a, f, \delta) = \sup{f(x) : x \in A and |x-a| < \delta},

m(a, f, \delta) = \inf{f(x) : x \in A and |x-a| < \delta}.

The oscillation o(f,a) of f at a is defined by o(f,a) = \lim\_{\delta \to 0} [M(a,f,\delta) – m(a,f,\delta)].

(Thm 1.10) The bounded function f is continuous at a iff o(f,a) = 0.

(Thm 11.1) Let A \subset \mathbf{R}^{n} be closed. If f : A \to \mathbf{R} is any bounded function, and \epsilon >0, then {x \in A : o(f,x) \ge \epsilon } is closed.

(Section 2) Differentiation

(subsection) Basic Definition

(Def) A function f : \mathbf{R}^{n} \to \mathbf{R}^{m} is differentiable at a \in \mathbf{R}^{n} if there is a linear transformation \lambda : \mathbf{R}^{n} \to \mathbf{R}^{m} s.t. \lim\_{h \to 0} \frac{|f(a+h) – f(a) - \lambda(h)| }{|h|} = 0.

The linear transformation \lambda is denoted Df(a) and called the derivative of f at a.

(Thm 2.1) If f : \mathbf{R}^{n} \to \mathbf{R}^{m} is differentiable at a \in \mathbf{R}^{n}, there is a unique linear transformation \lambda : \mathbf{R}^{n} \to \mathbf{r}^{m} s.t. \lim\_{h \to 0} \frac{|f(a+h) – f(a) - \lambda(h)| }{|h|} = 0.

(Def) The matrix of Df(a) : \mathbf{R}^{n} \to \mathbf{R}^{m} with respect to the unsual bases of \mathbf{R}^{n} and \mathbf{R}^{m}, this m \times n matrix is called the Jacobian matrix of f at a, and denoted f’(a).

(prop) a function f : \mathbf{R}^{n} \to \mathbf{R}^{m} to be differentiable on A if f is differentiable at a for each a \in A. If f : A \to \mathbf{R}^{m} , then f is called differentiable if f can be extended to a differentiable function on some open set containing A.

(subsection) Basic Theorems

(Chain Rule) (Thm 2.2) If f : \mathbf{R}^{n} \to \mathbf{R}^{m} is differentiable at a, and g : \mathbf{R}^{m} \to \mathbf{R}^{p} is differentiable at f(a), then the composition g \bullet f : \mathbf{R}^{n} \to \mathbf{R}^{p} is differentiable at a, and D(g \bullet f ) (a) = Dg(f(a)) \bullet Df(a).

(Thm 2.3)

If f : \mathbf{R}^{n} \to \mathbf{R}^{m} is a constant function (that is, if for some y \in \mathbf{R}^{m} we have f(x) = y for all x \in \mathbf{R}^{n}), then Df(a) = 0.

If f : \mathbf{R}^{n} \to \mathbf{R}^{m} is a linear transformation, then Df(a) = f.

If f : \mathbf{R}^{n} \to \mathbf{R}^{m}, then f is differentiable at a \in \mathbf{R}^{n{ iff each f^{i} is, and Df(a) = (Df^{1}(a), \cdots, Df^{m}(a)). Thus f’(a) is the m \times n matrix whose ith row is (f^{i})’(a).

If s : \mathbf{R}^{2} \to \mathbf{R} is defined by s(x,y) = x + y, then Ds(a,b) = s.

If p : \mathbf{R}^{2} \to \mathbf{R} is defined by p(x,y) = x \cdots y, then Dp(a,b)(x,y) = bx + ay. Thus p’(a,b) = (b,a).

(Cor 2.4) If f,g : \mathbf{R}^{n} \to \mathbf{R} are differentiable at a, then D(f+g)(a) = Df(a) + Dg(a), D(f \cdot g) (a) = g(a)Df(a) + f(a) Dg(a). If, moreover, g(a) \neq 0, then D(f/g)(a) = \frac{g(a)Df(a) – f(a)Dg(a)}{[g(a)]^{2}}.

(subsection) Partial Derivatives

(Def) If f : \mathbf{R}^{n} \to \mathbf{R} and a \in \mathbf{R}^{n} , the limit \lim\_{h \to 0} \frac{f(a^{1}, \cdots, a^{i} +h, \cdots, a^{n}) – f(a^{1}, \cdots, a^{n}) } {h} if it exists, is denoted D\_{i}f(a), and called the ith partial derivative of f at a.

(Def) If D\_{i}f(x) exists for all x \in \mathbf{R}^{n}, we obtain a function D\_{i}f : \mathbf{R}^{n} \to \mathbf{R} . The jth partial derivative of this function at x, that is, D\_{j}(D\_{i}f) (x) , is often denoted D\_{i,j} f(x).

(Thm 2.5) If D\_{i,j} f and D\_{j,i} f are continuous in an open set containing a, then D\_{i,j} f(a) = D\_{j,i} f(a). The function D\_{i,j} f is called a second-order (mixed) partial derivative of f.

(prop) the order of i\_{1}, \cdots, i\_{k} is immaterial in D\_{i1, \cdots, ik} f if f has continuous partial derivatives of all orders. A function with this property is called a C^{\infty} function.

(Thm 2.6) Let A \subset \mathbf{R}^{n}. If the maximum (or minimum) of f : A \to \mathbf{R} occurs at a point a in the interior of A and D\_{i}f(a) exists, then D\_{i}f(a) = 0.

(subsection) Derivatives

(Thm 2.7) If f : \mathbf{R}^{n} \to \mathbf{R}^{m} is differentiable at a, then D\_{j}f^{i}(a) exists for 1 \le i \le m, 1 \le j \le n and f’(a) is the m \times n matrix (D\_{j}f^{i}(a)).

(Thm 2.8) If f : \mathbf{R}^{n} \to \mathbf{R}^{m}, then Df(a) exists if all D\_{j}f^{i}(x) exists in an open set containing a and if each function D\_{j}f^{i} is continuous at a. Such a function f is called continuously differentiable at a.

(Thm 2.9) Let g\_{1} , \cdots, g\_{m} : \mathbf{R}^{n} \to \mathbf{R} be continuously differentiable at a, and let f : \mathbf{R}^{m} \to \mathbf{R} be differentiable at (g\_{1}(a), \cdots, g\_{m}(a)). Define the function F : \mathbf{R}^{n} \to \mathbf{R} by F(x) = f(g\_{1}(x), \cdots, g\_{m}(x)). Then D\_{i}F(a) = \sum\_{j = 1}^{m} D\_{j}f(g\_{1}(a), \cdots, g\_{m}(a)) \cdot D\_{i}g\_{j}(a).

(prop) for the function F : \mathbf{R}^{2} \to \mathbf{R} defined by F(x,y) = f(g(x,y), h(x),k(y)) where h,k : \mathbf{R} \to \mathbf{R}. Letting a = (g(x,y), h(x), k(y)), we obtain D\_{1}F(x,y) = D\_{1}f(a) \cdot D\_{1}g(x,y) + D\_{2} f(a) \cdot h’(a), D\_{2}F(x,y) = D\_{1}f(a) \cdot D\_{2}g(x,y) + D\_{3}f(a) \cdot k’(y).

(subsection) Inverse Functions

(Lem 2.10) Let A \subset \mathbf{R}^{n} be a rectangle and let f : A \to \mathbf{R}^{n} be continuously differentiable . If there is a number M s.t. |D\_{j}f^{i}(x) | \le M for all x in the interior of A< then |f(x) – f(y)| \le n^{2} M|x-y| for all x, y \in A.

(Inverse Function Theorem) (Thm 2.110 Suppose that f : \mathbf{R}^{n} \to \mathbf{R}^{n} is continuously differentiable in an open set containing a, and det f’(a) \neq 0. Then there is an open set V containing a and an open set W containing f(a) s.t. f : V \to W has a continuous inverse f^{-1} : W \to V which is differentiable and for all y \in W satisfies (f^{-1})’(y) = [f’(f^{-1}(y))]^{-1}.

an inverse function f^{-1} may exist even if det f’(a) = 0. However, if det f’(a) = 0, then f^{-1} cannot be differentiable at f(a).

(subsection) Implicit Functions

(Implicit Function Theorem) (Thm 2.12) Suppose f : \mathbf{R}^{n} \times \mathbf{R}^{m} \to \mathbf{R}^{m} is continuously differentiable in an open set containing (a,b) and f(a,b) = 0. Let M be the m \times m matrix (D\_{n+j}f^{i}(a,b)) 1 \le i,j \le m. If det M \neq 0, there is an open set A \subset \mathbf{R}^{n} containing a and an open set B \subset \mathbf{R}^{m} containing b, with the following property : for each x \in A there is a unique g(x) \in B s.t. f(x,g(x)) = 0. The function g is differentiable. These functions are said to be defined implicitly by the equation f(x,y) = 0.

(Thm 2.13) Let f : \mathbf{R}^{n} \to \mathbf{R}^{p} be continuously differentiable in an open set containing a, where p \le n. If f(a) = 0 and the p \times n matrix(D\_{j}f^{i}(a)) has rank p, then there is an open set A \subset \mathbf{R}^{n} containing a and a differentiable function h : A \to \mathbf{R}^{n} with differentiable inverse s.t. f \bullet h (x^{1}, \cdots, x^{n}) = (x^{n-p+1}, \cdots, x^{n}).

(section 3) Integration

(subsection) Basic definitions

(Def) a partition of a rectangle [a\_{1}, b\_{1}] \times \cdots \times [a\_{n}, b\_{n}] is a collection P = (P\_{1}, \cdots, P\_{n}) , where each P\_{i} is a partition of the interval [a\_{i},b\_{i}].

P = (P\_{1}, \cdots, P\_{n}) divides [a\_{1},b\_{1}] \times \cdots \times [a\_{n}, b\_{n}] into N = N\_{1}, \cdots, N\_{n} subrectangles. These subrectangles will be called subrectangles of the partition P.

(Def) Supose that A is a rectangle, f : A \to \mathbf{R} is a bounded function, and P is a partition of A. For each subrectangle S of the partition let m\_{S}(f) = \inf {f(x) : x \in S} , M\_{s}(f) = \sup{f(x) : x \in X}, and let v(S) be the volume of S. [The volume of a rectangle [a\_{1}, b\_{1}] \times \cdots \times [a\_{n}, b\_{n}] , and also of (a\_{1}, b\_{1}) \times \cdots \times (a\_{n}, b\_{n}) , is defined as (b\_{1} – a\_{1}) \cdot … \cdot (b\_{n} – a\_{m}) ] . the lower and upper sums of f for P are defined by

L(f,P) = \sum\_{S} m\_{S}(f) \cdot v(S) and U(f,P) = \sum\_{S} M\_{S}(f) \cdot v(S).

(Lem 3.1) Suppose the partition P’ refines P (that is, each subrectangle of P’ is contained in a subrectangle of P). Then L(f,P) \le L(f,P’) and U(f,P’) \le U(f,P) .

(Cor 3.2) If P and P’ are any two partitions, then L(f,P’) \le U(f,P).

The lease upper bound of all lower sums for f is less than or equal to the greatest lower bound of all upper sums for f.

(Def) A function f : A \to \mathbf{R} is called integrable on the rectangle A if f is bounded and \sup{L(f,P)} = \inf{U(f,P)} . This common number is then denoted \int\_{A} f, and called the integral of f over A. Often, the notation \int\_{A} f(x^{1}, \cdots, x^{n} ) dx^{1} \cdots dx^{n} is used.

(Thm 3-3) A bounded function f : A \to \mathbf{R} is integrable iff for every \epsilon >0 there is a partition P of A s.t. U(f,P) – L(f,P) < \epsilon.

(Prop)

Let f: A \to \mathbf{R} be a constant function, f(x) = c. \int\_{A} f = c \cdot v(A).

Let f:[0,1] \times [0,1] \to \mathbf{R} be defined by f(x,y) = \begin{cases} 0 & if x is rational, \\ 1 & if x is irrational.\end{cases} . f is not integrable.

(subsection) Measure Zero and Content Zero

(Def) A subset A of \mathbf{R}^{n} has (n-dimensional) measure 0 if for every \epsilon >0 there is a cover {U\_{1}, U\_{2}, U\_{3}, \cdots } of A by closed rectangles s.t. \sum\_{i = 1}^{\infty} v(U\_{i}) < \epsilon.

A set with only finitely many points has measure 0.

(Thm 3.4) If A = A\_{1} \cup A\_{2} \cup A\_{3} \cdots and each A\_{i} has measure 0, then A has measure 0.

(Def) A subset A of \mathbf{R}^{n} has (n-dimensional) content 0 if for every \epsilon>0 there is a finite cover {U\_{1}, \cdots, U\_{n} } of A by closed rectangles such that \sum\_{i = 1}^{n} v(U\_{i}) < \epsilon.

(Thm 3.5) If a < b, then [a,b] \subset \mathbf{R} does not have content 0. In fact, if {U\_{1}, \cdots, U\_{n} } is a finite cover of [a,b] by closed intervals, then \sum\_{i = 1}^{n} v(U\_{i}) \ge b-a.

(Thm 3.6) If A is compact and has measure 0, then A has content 0.

(subsection) Integrable functions

(Lem 3.7) Let A be a closed rectangle and let f : A \to \mathbf{R} be a bounded function s.t. o(f,x) < \epsilon for all x \in A. Then there is a partition P of A with U(f,P) – L(f,P) < \epsilon \cdot v(A).

(Thm 3.8) Let A be a closed rectangle and f : A \to \mathbf{R} a bounded function. Let B = {x : f is not continuous at x}, then f is integrable iff B is a set of measure 0.

(Def) If C \subset \mathbf{R}^{n}, the characteristic function \chi\_{C} of C is defined by \chi\_{C}(x) = \begin{cases} 0 & x \notin C, \\ 1 & x \in C .\end{cases}

If C \subset A for some closed rectangle A and f : A \to \mathbf{R} is bounded, then \int\_{C} f is defined as \int\_{A} f \cdot \chi\_{C}, provided f \cdot \chi\_{C} is integrable.

(Thm 3.9) The function \chi\_{C} : A \to \mathbf{R} is integrable iff the boundary of C has measure 0, (and hence content 0).

(Def) A bounded set C whose boundary has measure 0 is called Jordan-measurble. The integral \int\_{C} 1 is called (n-dimensional) content of C, or the (n-dimensional) volume of C. one-dimensional is often called length, and two-dimensional volume, area.

(subsection) Fubini’s Theorem

(Def) If f : A \to \mathbf{R} is a bounded function on a closed rectangle, then whether or not f is integrable, the least upper bound of all lower sums, and the greatest lower bound of all upper sums, both exist. They are called the lower and upper integrals of f on A, and denoted \mathbf{L} \int\_{A} f and \mathbf{U} \int\_{A} f.

(Fubini’s Theorem) (Thm 3.10) Let A \subset \mathbf{R}^{n} and B \subset \mathbf{R}^{m} be closed rectangles, and let f : A \times B \to \mathbf{R} be integrable. for x \in A let g\_{x} : B \to \mathbf{R} be defined by g\_{x} (y) = f(x,y) and let

\mathcal{L}(x) = \mathbf{L} \int\_{B} g\_{x} = \mathbf{L} \int\_{B} f(x,y) dy,

\mathcal{U}(x) = \mathbf{U} \int\_{B} g\_{x} = \mathbf{U} \int\_{B} f(x,y) dy.

Then \mathcal{L} and \mathcal{U} are integrable on A and

\int\_{A \times B} f = \int\_{A} \mathcal{L} = \int\_{A} (\mathbf{L} \int\_{B} f(x,y) dy) dx,

\int\_{A \times B} f = \int\_{A} \mathcal{U} = \int\_{A} (\mathbf{U} \int\_{B} f(x,y) dy) dx,

The integrals on the right side are called iterated integrals for f.

(Prop)

\int\_{A \times B} f = \int\_{A} (\int\_{B} f(x,y) dy) dx, if f is continuous.

In the case g\_{x} is not integrable for a finite number of x \in A, then \int\_{A \times B} f = \int\_{A} (\int\_{B} f(x,y) dy) dx, provided that \int\_{B} f(x,y) dy is defined arbitrarily, say as 0, when it does not exist.

Let f : [0,1] \times [0,1] \to \mathbf{R} be defined by f(x,y) = \begin{cases}1 & if x is irrational \\ 1 & if x is rational and y is irrational \\ 1 - \frac{1}{q} & if x = \frac{p}{q} in lowest terms and y is rational \end{cases} , Then f is integrable and \int\_{[0,1] \times [0,1]} f = 1. Now \int\_{0}^{1} f(x,y) dy = 1 if x is irrational, and does not exist if x is rational.

If A = [a\_{1}, b\_{1} ] \times \cdots \times [a\_{n}, b\_{n} ] and f : A \to \mathbf{R} is sufficiently nice, \int\_{A} f = \int\_{a\_{n}}^{b\_{n}} ( \cdots ( \int\_{a\_{1}}^{b\_{1}} f(x^{1} , \cdots, x^{n}) dx^{1}) \cdots ) dx^{n}.

If C \subset A \times B, Fubini’s theorem can be used to evaluate \int\_{C} f, since this is \int\_{A \times B} \chi\_{C} f.

(subsecton) Partitions of unity

(Thm 3.11) Let A \subset \mathbf{R}^{n} and let \mathcal{O} be an open cover of A. Then there is a collection \Phi| of C^{\infty} functions \phi defined in an open set containing A, with the following properties :

For each x \in A we have 0 \le \phi(x) \le 1.

For each x \in A there is an open set V containing x s.t. all but finitely many \phi \in \Phi are 0 on V.

For each x \in A we have \sum\_{\phi \in \Phi} \phi(x) = 1 ( by (2) for each x this sum is finite in some open set containing x)

For each \phi \in \Phi there is an open set U in \mathcal{O} s.t. \phi = 0 outside of some closed set containing in U.

A collection \Phi satisfying 1st and 3rd property is called a C^{\infty} partition of unity for A. If \Phi also satisfies 4th property , it is said to be subordinate to the cover \mathcal{O}.

(prop) Let C \subset A be compact. only finitely many \phi \in \Phi are not 0 on C.

(Def) An open cover \mathcal{O} of an open set A \subset \mathbf{R}^{n} is admissible if each U \in \mathcal{O} is contained in A. If \Phi is subordinate to \mathcal{O}, f : A \to \mathbf{R} is bounded in some open set around each point of A, and {x : f is discontinuous at x} has measure 0, then each \int\_{A} \phi \cdot |f| exists. we define f to be integrable (in the extended sense) if \sum\_{\phi \in \Phi} \int\_{A} \phi \cdot |f| converges.

(Thm 3.12) If \Psi is another partition of unity, subordinate to an admissible cover \mathcal{O}’ of A< then \sum\_{\psi \in \Psi} \int\_{A} \psi \cdot |f| also converges, and \sum\_{\phi \in \Phi} \int\_{A} \phi \cdot f = \sum\_{\psi \in \Psi} \int\_{A} \psi \cdot f.

If A and f are bounded, then f is integrable in the extended sense.

If A is Jordan-measurable and f is bounded, then this definition of \int\_{A} f agrees with the old one.

(subsection) Change of Variable)

(Prop) If g : [a,b] \to \mathbf{R} is continuously differentiable and f : \mathbf{R} to \mathbf{R} is continuous, then, \int\_{g(a)}^{g(b)} f = \int\_{a}^{b} (f \bullet g) \cdot g’ .

(Thm 3.13) Let A \subset \mathbf{R}^{n} be an open set and g : A \to \mathbf{R}^{n} a 1-1, continuously differentiable function s.t. det g’(x) \neq 0 for all x \in A. If f: g(A) \to \mathbf{R} is integrable, then \int\_{g(A)} f = \int\_{A} (f \bullet g) |det g’ | .

(Sard’s Theorem) (Thm 3.14) Let g : A \to \mathbf{R}^{n} be continuously differentiable, where A \subset \mathbf{R}^{n} is open, and let B = { x \in A : det g’(x) = 0} . Then g(B) has measure 0.

(Section 4) Integration on Chains

(subsection) Algebraic Preliminaries

(Def) If V is a vector space (over \mathbf{R}), we will denote the k-fold product V \times \cdots \times V by V^{k} .

A function T : V^{k} \to \mathbf{R} is called multilinear if for each i with 1 \le i \le k we have T(v\_{1} , \cdots , v\_{i} + v\_{i}’ , \cdots , v\_{k}) = T(v\_{1} , \cdots, v\_{i} , \cdots, v\_{k} ) + T(v\_{1} , \cdots, v\_{i}’ , \cdots, v\_{k} ) , T(v\_{1}, \cdots, av\_{i} , \cdots, v\_{k}) = aT(v\_{1}, \cdots, v\_{i} , \cdots, v\_{k}) .

(Def) A multilinear function T : V^{k} \to \mathbf{R} is called a k-tensor on V and the set of all k-tensors, denoted \mathcal{J}^{k} (V) and a \in \mathbf{R} we define (S+T) (v\_{1}, \cdots, v\_{k}) = S(v\_{1}, \cdots, v\_{k}) + T(v\_{1}, \cdots, v\_{k}) , (aS)(v\_{1}, \cdots, v\_{k}) = a \cdot S(v\_{1} , \cdots, v\_{k} ) .

(Def) If S \in \mathcal{J}^{k} (V) and T \in \mathcal{J}^{l} (V) , we define the tensor product S \otimes T \in \mathcal{J}^{k+1} (V) by S \otimes T (v\_{1}, \cdots, v\_{k} , v\_{k+1}, \cdots, v\_{k+l}) = S(v\_{1}, \cdots, v\_{k}) \cdot T(v\_{k+1}, \cdots, v\_{k+l}) .

(prop)

(S\_{1} + S\_{2}) \otimes T = S\_{1} \otimes T + S\_{2} \otimes T,

S \otimes (T\_{1} + T\_{2}) = S \otimes T\_{1} + S \otimes T\_{2},

(aS) \otimes T = S \otimes (aT) = a(S \otimes T),

(S \otimes T) \otimes U = S \otimes (T \otimes U).

(Thm 4.1) Let v\_{1} , \cdots, v\_{n} be a basis for V, and let \phi\_{1} , \cdots, \phi\_{n} be the dual basis , \phi\_{i} (v\_{j}) = \delta\_{ij}. Then the set of all k-fold tensor products \phi\_{i\_{1}} \otimes \cdots \otimes \phi\_{i\_{k}} 1 \le i\_{1}, \cdots, i\_{k} \le n is a basis for \mathcal{J}^{k} (V) , which therefore has dimension n^{k}.

(Def) If f : V \to W is a linear transformation, a linear transformation f^{\*} : \mathcal{J}^{k} (W) \to \mathcal{J}^{k}(V) is defined by f^{\*}T(v\_{1}, \cdots, v\_{k}) = T(f(v\_{1}) , \cdots, f(v\_{k})) for T \in \mathcal{J}^{k}(W) and v\_{1}, \cdots, v\_{k} \in V. It is easy to verify that f^{\*} (S \otimes T) = f^{\*}S \otimes f^{\*}T.

(Def) Inner product on V to be a 2-tensor T such that T is symmetric, that is T(v,w) = T(w,v) for v,w \in V and such that T is positive definite, that is, T(v,v) >0 if v \neq 0. We distinguish \langle , \rangle as the usual inner product on \mathbf{R}^{n}.

(Thm 4.2) If T is an inner product on V, there is a basis v\_{1}, \cdots, v\_{n} for V s.t. T(v\_{i}, v\_{j}) = \delta\_{ij} . (Such a basis is called orthonormal with respct to T.) Consequently there is an isomorphism f : \mathbf{R}^{n} \to V s.t. T(f(x), f(y)) = \langle x, y \rangle for x, y \in \mathbf{R}^{n}. In other words f^{\*}T = \langle , \rangle.

(Def) A k-tensor \omega \in \mathcal{J}^{k}(V) is called alternating if \omega(v\_{1}, \cdots, v\_{i}, \cdots, v\_{j}, \cdots, v\_{k}) = - \omega(v\_{1}, \cdots, v\_{j}, \cdots, v\_{i}, \cdots, v\_{k}) for all v\_{1}, \cdots, v\_{k} \in V.

The set of all alternating k-tensors is clearly a subspace \Lambda^{k}(V) of \mathcal{J}^{k}(V).

(Def) the sign of a permutation \sigma, denoted \sgn \sigma, is +1 if \sigma is even and -1 if \sigma is odd. If T \in \mathcal{J}^{k}(V), we define Alt(T) by Alt(T)(v\_{1}, \cdots, v\_{k}) = \frac{1}{k!} \sum\_{\sigma \in S\_{k}} \sgn \sigma \cdot T(v\_{\sigma (1)} , \cdots, v\_{\sigma(k)} ), where S\_{k} is the set of all permutations of the numbers 1 to k.

(Thm 4.3)

If T \in \mathcal{J}^{k}(V), then Alt (T) \in \Lambda^{k}(V).

If \omega \in \Lambda^{k} (V), then Alt(\omega) = \omega.

If T \in \mathcal{J}^{k} (V), then Alt(Alt(T)) = Alt(T).

(Def) The wedge product \omega \wedge \eta \in \Lambda^{k+l} (V) by \omega \wedge \eta = \frac{(k+l)!}{k!l!} Alt(\omega \otimes \eta).

(prop) (\omega\_{1} + \omega\_{2}) \wedge \eta = \omega\_{1} \wedge \eta + \omega\_{2} \wedge \eta,

\omega \wedge (\eta\_{1} + \eta\_{2} ) = \omega \wedge \eta\_{1} + \omega \wedge \eta\_{@},

a \omega \wedge \eta = \omega \wedge a \eta = a(\omega \wedge \eta),

\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega,

f^{\*}(\omega \wedge \eta) = f^{\*} (\omega) \wedge f^{\*} (\eta)

(Thm 4.4)

If S \in \mathcal{J}^{k}(V) and T \in \mathcal{J}^{l}(V) and Alt(S) = 0, then Alt(S \otimes T) = Alt(T \otimes S ) = 0.

Alt(Alt(\omega \otimes \eta) \otimes \theta) = Alt(\omega \otimes \eta \otimes \theta) = Alt(\omega \otimes Alt(\eta \otimes \theta)).

If \omega \in \Lambda^{k}(V), \eta \in \Lambda^{l}(V), and \theta \in \Lambda^{m} (V), then (\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k + l + m)!}{k!l!m!} Alt(\omega \otimes \eta \otimes \theta) .

(Thm 4.5) If v\_{1}, \cdots , v\_{n} is a basis for V and \phi\_{1}, \cdots, \phi\_{n} is the dual basis, The set of all \phi\_{i\_{1}} \wedge \cdots \wedge \phi\_{i\_{k}} 1 \le i\_{1}, < i\_{2} < \cdots < i\_{k} \le n is a basis for \Lambda^{k} (V), which therefore has dimension \binom{n}{k} = \frac{n!}{k!(n-k)!}.

If V has dimension n, \Lambda^{n} (V) has dimension 1. Thus all alternating n-tensors on V are multiples of any non-zero one.

(Thm 4.6) Let v\_{1}, \cdots, v\_{n} be a basis for V, and let \omega \in \Lambda^{n} (V). If w\_{o} = \sum\_{j = 1}^{n} a\_{ij}v\_{j} are n vectors in V, then \omega(w\_{1}, \cdots, w\_{n}) = det(a\_{ij}) \cdot \omega(v\_{1}, \cdots, v\_{n}) .

(Def) a nonzero \omega \in \Lambda^{n}(V) splits the bases of V into two disjoint groups, those with \omega(v\_{1}, \cdots, v\_{n}) > 0 and those for which \omega(v\_{1}, \cdots, v\_{n}) < 0 ; this criterion is independent of \omega and can always be used to divide the bases of V into two disjoint groups. Either of these two groups is called an orientation for V.

The orientation to which a basis v\_{1}, \cdots, v\_{n} belongs is denoted [v\_{1}, \cdots, v\_{n}] and the other orientation is denoted –[v\_{1}, \cdots, v\_{n} ] . In \mathbf{R}^{n} we define the usual orientation as [e\_{1}, \cdots, e\_{n} ] .

(Def) For a general vector space V, if an inner product T for V is given. If an orientation \mu for V has also ben given, it follows that there is a unique \omega \in \Lambda^{n} (V) such that \omega(v\_{1}, \cdots, v\_{n}) = 1 whenever v\_{1}, \cdots, v\_{n} is an orthonormal basis such that [v\_{1}, \cdots, v\_{n}] = \mu. This unique \omega is called the volume element of V, determined by the inner product T and orientation \mu.

(Def) If v\_{1}, \cdots, v\_{n-1} \in \mathbf{R}^{n} and \phi is defined by \phi(w) = det \begin{pmatrix} v\_{1} \\ \vdots \\ v\_{n-1} \\ w \end{pmatrix}. then \phi \in \Lambda^{1}(\mathbf{R}^{n}) ; therefore there is a unique z \in \mathbf{R}^{n} s.t. \langle w , z \rangle = \phi (w) = det \begin{pmatrix} v\_{1} \\ \vdots \\ v\_{n-1} \\ w \end{pmatrix} . This z is denoted v\_{1} \times \cdots \times v\_{n-1} and called the cross product of v\_{1}, \cdots, v\_{n-1} .

(prop) v\_{\sigma(1)} \times \cdots \times v\_{\sigma(n-1)} = \sgn \sigma \cdot v\_{1} \times \cdots \times v\_{n-1},

v\_{1} \times \cdots \times av\_{i} \times \cdots \times v\_{n-1} = a \times (v\_{1} \times \cdots \times v\_{n-1} ) ,

v\_{1} \times \cdots \times (v\_{i} + v\_{i} ‘) \times \cdots \times v\_{n-1} = v\_{1} \times \cdots \times v\_{i} \times \cdots \times v\_{n-1} + v\_{1} \times \cdots \times v\_{i}’ \times \cdots \times v\_{n-1} .

(subsection) Fields and Forms

(Def) If p \in \mathbf{R}^{n} , the sets of all pairs (p,v) , for v \in \mathbf{R}^{n} , is denoted \mathbf{R}^{n}\_{p}, and called the tangent space of \mathbf{R}^{n} at p. This set is made into a vector space by (p,v) + (p,w) = (p , v+w) , a \cdot (p,v) = (p, av).

define p + v to be the end point of (p,v). write (p,v) as v\_{p} .

(Def) Usual inner product \langle , \rangle\_{p} for \mathbf{R}^{n}\_{p} is defined by \langle v\_{p} , w\_{p} \rangle\_{p} = \langle v , w \rangle, and the usual orientation for \mathbf{R}^{n}\_{p} is [(e\_{1})\_{p} , \cdots , (e\_{n})\_{p}] .

(Def) A vector field is a function F s.t. F(p) \in \mathbf{R}^{n}\_{p} for each p \in \mathbf{R}^{n} . For each p there are numbers F^{1}(p) , \cdots, F^{n} (p) s.t. F(p) = F^{1}(p) \cdot (e\_{1})\_{p} + \cdots + F^{n}(p) \cdot (e\_{n})\_{p} . We obtain n component functions F^{i} : \mathbf{R}^{n} \to \mathbf{R}.

Vector field F is called continuous, differentiable, etc., if the functions F^{i} are.

Operations on vectors yield operations on vector fields when applied at each point separately.

(Def) divergence, div F of F, as \sum\_{i = 1}^{n} D\_{i}F^{i}. \nabla = \sum\_{i = 1}^{n} D\_{i} \cdot e\_{i} , div(F) = \langle \nabla , F \rangle .

(Def) If n = 3 we write (\nabla \times F) (p) = (D\_{2} F^{3} – D\_{3} F^{2} )(e\_{1})\_{p} + (D\_{3} F^{1} – D\_{1}F^{3})(e\_{2})\_{p} + (D\_{1} F^{2} – D\_{2}F^{1})(e\_{3})\_{p} . The vector field \nabla \times F is called curl F.

(Def) a function \omega with \omega(p) \in \Lambda^{k}(\mathbf{R}^{n}\_{p}) . such a function is called a k-form on \mathbf{R}^{n}, or a differential form. If \phi\_{1}(p) , \cdots, \phi\_{n}(p) is the dual basis to (e\_{1})\_{p} , \cdots, (e\_{n})\_{p}, then \omega(p) = \sum\_{i\_{1} < \cdots < i\_{k}} \omega\_{i\_{1}, \cdots, i\_{k}} (p) \cdot [\phi\_{i\_{1}} (p) \wedge \cdots \wedge \phi\_{i\_{k}}(p)] for certain functions \omega\_{i\_{1}, \cdots, i\_{k}} .

The form \omega is called continuous, differentiable, etc., if these functions are.

The sum \omega + \eta, product f \cdot \omega, and wedge product \omega \wedge \eta are defined in the obvious way.

(Def) If f : \mathbf{R}^{n} \to \mathbf{R} is differentiable, then Df(p) \in \Lambda^{1} (\mathbf{R}^{n} . we obtain a 1-form df, defined by df(p)(v\_{p}) = Df(p)(v).

(Def) Let x^{i} denote the function \pi\_{i}. Since dx^{i}(p)(v\_{p}) = d\pi^{i} (p)(v\_{p}) = D\pi^{i} (p)(V) = v^{i} , dx^{1}(p), \cdots, dx^{n}(p) is just the dual basis to (e\_{1})\_{p} , \cdots, (e\_{n})\_{p} . Thus every k-form \omega can be written \omega = \sum\_{i\_{1} < \cdots < i\_{k}} \omega\_{i\_{1}, \cdots, i\_{k}} dx^{i\_{1}} \wedge \cdots dx^{i\_{k}}

(Thm 4.7) If f : \mathbf{R}^{n} \to \mathbf{R} is differentiable, then df = D\_{1}f \cdot dx^{1} + \cdots + D\_{n}f \cdot dx^{n}. In classical notation, df = \frac{\partial f}{\partial x^{1}} dx^{1} + \cdots + \frac{\partial f }{\partial x^{n}} dx^{n}.

(Def) Consider now a differentiable function f : \mathbf{R}^{n} \to \mathbf{R} we have a linear transformation Df(p) : \mathbf{R}^{n} \to \mathbf{R}^{m} . we produce linear transformation f\_{\*} : \mathbf{R}^{n}\_{p} \to \mathbf{R}^{m}\_{f(p)} defined by f\_{\*}(v\_{p}) = (Df(p)(v))\_{f(p)}. This induces a linear transformation f^{\*} : \Lambda^{k}(\mathbf{R}^[m]\_{f(p)}) \to \Lambda^{k}(\mathbf{R}^{n}\_{p}) . If \omega is a k-form on \mathbf{R}^{m} we can therefore define a k-form f^{\*} \omega on \mathbf{R}^{n} by (f^{\*} \omega) (p) = f^{\*}(\omega(f(p))).

This means that if v\_{1}, \cdots, v\_{k} \in \mathbf{R}^{n}\_{p}, then we have f^{\*} \omega(p) (v\_{1}, \cdots, v\_{k}) = \omega(f(p))(f\_{\*}(v\_{1}, \cdots, f\_{\*}(v\_{k})).

(Thm 4.8) If f: \mathbf{R}^{n} \to \mathbf{R}^{m} is differentiable, then

f^{\*}(dx^{i}) = \sum\_{j = 1}^{n} D\_{j}f^{i}\cdot dx^{j} = \sum\_{j = 1}^{n} \frac{\partial f^{i}}{\partial x^{j}} dx^{j}.

f^{\*} (\omega\_{1} + \omega\_{2}) = f^{\*}(\omega\_{1}) + f^{\*} (\omega\_{2}) .

f^{\*} (g \cdot \omega) = (g \bullet f) \cdot f^{\*} \omega.

f^{\*} (\omega \wedge \eta) = f^{\*} \omega \wedge f^{\*} \eta.

(Thm 4.9) If f : \mathbf{R}^{n} \to \mathbf{R}^{n} is differentiable, then f^{\*} (h dx^{1} \wedge \cdots \wedge dx^{n}) = (h \bullet f)(det f’) dx^{1} \wedge \cdots \wedge dx^{n}.

(Def) If \omega = \sum\_{i\_{1} < \cdots < i\_{k}} \omega\_{i\_{1}, \cdots, i\_{k}} dx^{i\_{1}} \wedge \cdots dx^{i\_{k}} , we define a (k+1)-form d\omega, the differential of \omega, by d\omega = \sum\_{i\_{1} < \cdots < i\_{k}} d \omega\_{i\_{1}, \cdots, i\_{k}} \wedge dx^{i\_{1}} \wedge \cdots dx^{i\_{k}} = \sum\_{i\_{1} < \cdots < i\_{k}} \sum\_{\alpha = 1}^{n} D\_{\alpha}(\omega\_{i\_{1}, \cdots, i\_{k}} ) \cdot dx^{\alpha} \wedge dx^{i\_{1}} \wedge \cdots dx^{i\_{k}}

(Thm 4.10)

d(\omega + \eta) = d\omega + d\eta.

If \omega is a k-form and \eta is an l-form, then d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{k} \omega \wedge d\eta.

d(d\omega)) = 0. Briefly, d^{2} = 0.

If \omega is a k-form on \mathbf{R}^{m} and f : \mathbf{R}^{n} \to \mathbf{R}^{m} is differentiable, then f^{\*} (d\omega) = d(f^{\*} \omega).

(Def) a form \omega is called closed if d \omega = 0 and exact if \omega = d \eta, for some \eta.

(Def) An open set A \subset \mathbf{R}^{n} with the properyty that whenever x \in A, the line segment from 0 to x is contained in A; such an open set is called star-shaped with respect to 0.

(Poincare Lemma) (Thm 4.11) If A \subset \mathbf{R}^{n} is an open set star-shaped with respect to 0, then every closed form on A is exact.

(subsection) Geometric preliminaries

(Def) A singular n-cube in A \subset \mathbf{R}^{n} is a continuus function c : [0,1]^{n} \to A (here [0,1]^{n} denotes the n-fold product [0,1] \times \cdots \times [0,1]). \mathbf{R}^{0} and [0,1]^{0} both denote {0}.

A singular 1-cube is often called a curve.

The standard n-cube I^{n} : [0,1]^{n} \to \mathbf{R}^{n} defined by I^{n} (x) = x for x \in [0,1]^{n}.

(Def) a finite sum of singular n-cubes with integer coefficients is called an n-chain in A. For each singular n-chain c in A we shall define an (n-1)-chain in A called the boundary of c and denoted \partial c.

(Def) For each i with 1 \le i \le n we define two singular (n-1)-cubes I^{n}\_{(i,0)} and I^{n}\_{(i,1)} as follows. If x \in [0,1]^{n-1}, then

I^{n}\_{(i,0)} (x) = I^{n}(x^{1} , \cdots, x^{i-1}, 0, x^{i}, \cdots, x^{n-1}) = (x^{1} , \cdots, x^{i-1}, 0, x^{i}, \cdots, x^{n-1}),

I^{n}\_{(i,1)} (x) = I^{n}(x^{1} , \cdots, x^{i-1}, 1, x^{i}, \cdots, x^{n-1}) = (x^{1} , \cdots, x^{i-1}, 1, x^{i}, \cdots, x^{n-1}).

We call I^{n}\_{(i,0)} the (i,0)-face of I^{n} and I^{n}\_{(i,1)} the (i,1)-face.

we then define \partial I^{n} = \sum\_{i = 1}^{n} \sum\_{\alpha = 0,1} (-1)^{i+\alpha} I^{n}\_{(i + \alpha)} .

(Def) For a general singular n-cube c: [0,1]^{n} \to A we define the (i,\alpha) -face, c\_{(i,\alpha)} = c \bullet (I^{n}\_{(i,\alpha)}) and then define \partial c = \sum\_{i = 1}^{n} \sum\_{\alpha = 0,1} (-1)^{i+\alpha} c\_{(i,\alpha)}.

We define the boundary of an n-chain \sum a\_{i}c\_{i} by \partial (\sum a\_{i} c\_{i}) = \sum a\_{i} \partial(c\_{i}) .

(Thm 4.12) If c is an n-chain in A, then \partial(\partial c) = 0. Briefly, \partial^{2} = 0.

(subsection) The fundamental theorem of calculus

(Def) If \omega is a k-form on [0,1]^{k}, then \omega = f dx^{1} \wedge \cdots \wedge dx^{k} for a unique function f. we define \int\_{[0,1]^{k}} \omega = \int\_{[0,1]^{k}} f. we could also write this as \int\_{[0,1]^{k}} f dx^{1} \wedge \cdots \wedge dx^{k} = \int\_{[0,1]^{k}} f(x^{1}, \cdots, x^{k}) dx^{1} \cdots dx^{k} . If \omega is a k-form on A and c is a singular k-cube in A, we define \int\_{c} \omega = \int\_{[0,1]^{k}} c^{\*} \omega.

(Def) If c : {0} \to A is a singular 0-cube in A we define \int\_{c} \omega = \omega(c(0)) . The integral of \omega over a k-chain c = \sum a\_{i} c\_{i} is defined by \int\_{c} \omega = \sum a\_{i} \int\_{c{i}} \omega. The integral of a 1-form over a 1-chain is often called a line integral.

(prop) If P dx + Q dy is a 1-form on \mathbf{R}^{2} and c:[0,1] \to \mathbf{R}^{2} is a singular 1-cube,

\int\_{c} P dx + Q dy = \lim \sum\_{i = 1}^{n} [c^{1} (t\_{i}) – c^{1}(t\_{i-1})] \cdot P(c(t^{i})) + [c^{2} (t\_{i}) – c^{2}(t\_{i-1})] \cdot Q(c(t^{i}))

where t\_{0}, \cdots, t\_{n} is a partition of [0,1], the choice of t^{i} in [t\_{i-1}, t\_{i}] is arbitrary, and the limit is taken over all partitions as the maximum of |t\_{i} – t\_{i-1}| goes to 0.

(Def) Analogous definitions for surface integrals, integrals of 2-forms over singular 2-cubes.

(Stoke’s Theorem) (Thm 4.13) if \omega is a (k-1)-form on an open set A \subset \mathbf{R}^{n} and c is a k-chain in A, then \int\_{c} d\omega = \int\_{\partial c} \omega.

(Section 5) Integration on Manifolds

(subsection) Manifolds

(Def) If U and V are open sets in \mathbf{R}{n}, a differentiable function h : U \to V with a differentiable inverse h^{-1} : V \to U will be called a diffeomorphism.

‘Differentiable’ henceforth means C^{\infty}.

(Def) A subset M of \mathbf{R}^{n} is called a k-dimensional manifold (in \mathbf{R}^{n}) if for every point x \in M the following condition is satisfied :

(M) There is an open set U containing x, an open set V \subset \mathbf{R}^{n}, and a diffeomorphism h : U \to V s.t. h(U \cap M) = V \cap (\mathbf{R}^{k} \times {0}) = {y \in V : y^{k+1} = \cdots = y^{n} = 0}.

We say U \cap M is up to diffeomorphism \mathbf{R}^{k} \times {0}.

(Def) n-sphere S^{n}, defined as {x \in \mathbf{R}^{n+1} : |x| = 1}.

(Thm 5.1) Let A \subset \mathbf{R}^{n} be open and let g : A \to \mathbf{R}^{p} be a differentiable function s.t. g’(x) has rank p whenever g(x) = 0. Then g^{-1} (0) is an (n-p)-dimensional manifold in \mathbf{R}^{n}.

(Thm 5.2) A subset M of \mathbf{R}^{n} is a k-dimensional manifold iff for each point x \in M the following coordinate condition is satisfied :

(C) There is an open set U containing x, an open set W \subset \mathbf{R}^{k}, and a 1-1 differentiable function f : W \to \mathbf{R}^{n} s.t. f(W) = M \cap U, f’(y) has a rank k for each y \in W, f^{-1} : f(W) \to W is continuous.

such a function f is called a coordinate system around x.

(Def) The half-space H^{k} \subset \mathbf{R}^{k} is defined as {x \in \mathbf{R}^{k} : x^{k} \ge 0}. A subset M of \mathbf{R}^{n} is a k-dimensional manifold-with-boundary if for every point x \in M either condition (M) or the following condition is satisfied:

(M’) There is an open set U containing x, an open set V \subset \mathbf{R}^{n}, and a diffeomorphism h : U \to V s.t. h(U \cap M) = V \cap (H^{k} \times {0}) = {y \in V : y^{k} \ge 0 and y^{k+1} = \cdots = y^{n} = 0}

and h(x) has kth component 0.

(Def) The set of all points x \in M for which condition M’ is satisfied is called the boundary of M and denoted \partial M.

(subsection) Fields and forms on Manifolds

(Def) Let M be a k-dimensional manifold in \mathbf{R}^{n} and let f : W \to \mathbf{R}^{n} be a coordinate system around x = f(a). Since f’(a) has rank k, the linear transformation f\_{\*} : \mathbf{R}^{k}\_{a} \to \mathbf{R}^{n}\_{x} is 1-1, and f\_{\*}(\mathbf{R}^{k}\_{a}) is a k-dimensional subspace of \mathbf{R}^{n}\_{x}. If g: V \to \mathbf{R}^{n} is another coordinate system, with x = g(b), then g\_{\*} (\mathbf{R}^{k}\_{b}) = f\_{\*}(f^{-1} \bullet g)\_{\*}(\mathbf{R}^{k}\_{b}) = f\_{\*}(\mathbf{R}^{k}\_{a}). Thus the k-dimensional subspace f\_{\*}(\mathbf{R}^{k}\_{b}) does not depend on the coordinate system f. This subspace is denoted M\_{x}, and is called the tangent space of M at x.

a natural inner product T\_{x} on M\_{x}, induced by that on \mathbf{R}^{n}\_{x} : if v,w \in M\_{x} define T\_{x}(v,w) = \langle v,w \rangle\_{x}.

(prop) Suppose that A is an open set containing M, and F is differentiable vector field on A s.t. F(x) \in M\_{x} for each x \in M. If f: W \to \mathbf{R}^{n} is a coordinate system, there is a unique (differentiable) vector field G on W s.t. f\_{\*}(G(a)) = F(f(a)) for each a \in W.

(Def) Consider a function F which merely assigns a vector F(x) \in M\_{x} for each x \in M ; such a function is called a vector field on M.

define F to be differentiable if G corresponding to F is differentiable.

(Def) a function \omega which assigns \omega(x) \in \Lambda^{p}(M\_{x}) for each x \in M is called a p-form on M.

If f : W \to \mathbf{R}^{n} is a coordinate system, then f^{\*} \omega is a p-form on W; we define \omega to be differentiable if f^{\*} \omega is.

(Thm 5.3) There is a unique (p+1)-form d\omega on M s.t. for every coordinate system f : W \to \mathbf{R}^{n} we have f^{\*}(d\omega) = d(f^{\*} \omega).

(Def) when choosing an orientation \mu\_{x} for each tangent space M\_{x} of a manifold M, such choices are called consistent provided that for every coordinate system f : W \to \mathbf{R}^{n} and a,b \in W the relation [f\_{\*} ((e\_{1})\_{a}), \cdots, f\_{\*}((e\_{k})\_{a})] = \mu\_{f(a)} holds iff [f\_{\*} ((e\_{1})\_{b}), \cdots, f\_{\*}((e\_{k})\_{b})] = \mu\_{f(b)}.

(Def) Suppose orientations \mu\_{x} have been chosen consistently. If f: W \to \mathbf{R}^{n} is a coordinate system s.t. [f\_{\*} ((e\_{1})\_{a}), \cdots, f\_{\*}((e\_{k})\_{a})] = \mu\_{f(a)} for one, and hence for every a \in W, then f is called orientation-preserving.

If f is not orientation-preserving and T: \mathbf{R}^{k} \to \mathbf{R}^{k} is a linear transformation with det T = -1, then f \bullet T is orientation-preserving. Therefore there is an orientation-preserving coordinate system around each point.

(Def) A manifold for which orientations \mu\_{x} can be chosen consistently is called orientable, and a particular choice of the \mu\_{x} is called an orientation \mu of M. A manifold together with an orientation \mu is called an oriented manifold.

(prop) If M is a k-dimensional manifold-with-boundary and x \in \partial M, then (\partial M)\_{x} is a (k-1)-dimensional subspace of the k-dimensional vector space M\_{x}. Thus there are exactly two unit vectors in M\_{x} which are perpendicular to (\partial M)\_{x}.

(Def) If f : W \to \mathbf{R}^{n} is a coordinate system with W \subset H^{k} and f(0) = x, then only one of these unit vectors is f\_{\*}(v\_{0}) for some v\_{0} with v^{k} <0. This unit vector is called the outward unit normal n(x).

(Def) Suppose that \mu is an orientation of a k-dimensional manifold-with-boundary M. if M is orientable, \parital M is also orientable, and an orientation \mu for M determines an orientation \partial \mu for \partial M, called the induced orientation.

If we apply these definitions to H^{k} with the usual orientation, we find that the induced orientation on \mathbf{R}^{k-1} = {x \in H^{k} : x^{k} = 0} is (-1)^{k} times the usual orientation.

(Def) If M is an oriented (n-1)-dimensional manifold in \mathbf{R}^{n}, a substitute for outward unit normal can be defined. If [v\_{1}, \cdots, v\_{n-1}] = \mu\_{x} , we choose n(x) in \mathbf{R}^{n}\_{x} so that n(x) is a unit vector perpendicular to M\_{x} and [n(x), v\_{1}, \cdots, v\_{n-1}] is the usual orientation of \mathbf{R}^{n}\_{x} . We still call n(x) the outward unit normal to M (determined by \mu).

Conversely, if a continuous family of unit normal vectors n(x) is defined on all of M, then we can determine the orientation of M.

(subsection) Stoke’s Theorem on Manifolds

(Def) If \omega is a p-form on a k-dimensional manifold-with-boundary M and c is a singular p-cube in M, we define \int\_{c} \omega = \int\_{[0,1]^{p}} c^{\*} \omega.

In the case p = k there is an open set W \supset [0,1]^{k} and a coordinate system f : W \to \mathbf{R}^{n} s.t. c(x) = f(x) for x \in [0,1]^{k};

if M is oriented, the singular k-cube c is called orientation-preserving if f is.

(Thm 5.4) If c\_{1}, c\_{2} : [0,1]^{k} \to M are two orientation-preserving singular k-cubes in the oriented k-dimensional manifold M and \omega is a k-form on M s.t. \omega = 0 outside of c\_{1}([0,1]^{k}) \cap c\_{2}([0,1]^{k}), then \int\_{c\_{1}} \omega = \int\_{c\_{2}} \omega.

(Def) Let \omega be a k-form on an oriented k-dimensional manifold M. If there is an orientation-preserving singular k-cube c in M s.t. \omega = 0 outside of c([0,1]^{k}) , we define \int\_{m} \omega = \int\_{c} \omega.

\int\_{M} \omega does not depend on the choice of c.

(prop) Suppose noe that \omega is an arbitrary k-form on M. There is an open cover \mathcal{O} of M s.t. for each U \in \mathcal{O} there is an orientation-preserving singular k-cube c with U \subset c([0,1]^{k}) .

(Def) Let \Phi be a partition of unity for M subordinate to this cover. We define \int\_{M} \omega = \sum\_{\phi \in \Phi} \int\_{m} \phi \cdot \omega provided the sum converges as describes in the discussion preceding Theorem 3-12.

(Def) All our definitions could have been given for a k-dimensional manifold-with-boundary M with orientation \mu.

(Stokes’ Theorem) (Thm 5.5) If M is a compact oriented k-dimensional manifold-with-boundary and \omega is a (k-1)-form on M, then \int\_{m} d\omega = \int\_{\partial M} \omega. (Here \partial M is given the induced orientation).

(subsection) The Volume element

(Def) Let M be a k-dimensional manifold (or manifold-with-boundary) in \mathbf{R}^{n}, with an orientation \mu. If x \in M, then \mu\_{x} and the inner product T\_{x} we defined previously determine a volume element \omega(x) \in \Lambda^{k}(M\_{x}) . We obtain a nowhere-zero k-form \omega on M, which is called the volume element on M (determined by \mu) and denoted dV.

(Def) the volume of M is defined as \int\_{M} dV, provided this integral exists, which is certainly the case if M is compact.

volume is called length or surface area for one-and two-dimensional manifolds, and dV is denoted ds (the element of length) or dA (or dS, the element of surface area).

(Thm 5.6) Let M be an oriented two-dimensional manifold (or manifold-with-boundary) in \mathbf{R}^{3} and let n be the unit outward normal. Then

dA = n^{1} dy \wedge dz + n^{2} dz \wedge dx + n^{3} dx \wedge dy.

Moreover, on M we have

n^{1} dA = dy \wedge dz.

n^{2} dA = dz \wedge dx.

n^{3} dA = dx \wedge dy.

(prop) If c : [0,1] to \mathbf{R}^{n} is differentiable and c([0,1]) is a one-dimensional manifold-with-boundary, the length of c([0,1]) is the least upper bound of the lengths of inscribed broken lines. For c:[0,1]^{2} \to \mathbf{R}^{n}, it is not true.

(subsection) The classical theorems

(Green’s Theorem) (Thm 5.7) Let M \subset \mathbf{R}^{2} be a compact two-dimensional manifold-with-boundary. Suppose that \alpha, \beta : M \to \mathbf{R} are differentiable. Then \int\_{\partial M} \alpha dx + \beta dy = \int\_{M} (D\_{1} \beta – D\_{2} \alpha) dx \wedge dy = \int\_{M} \int (\frac{\partial \beta}{\partial x}- \frac{\parital \alpha }{\partial y}) dx dy.

(Divergence Theorem) (Thm 5.8) Let M \subset \mathbf{R}^{3} be a compact three-dimensional manifold-with-boundary and n the unit outward normal on \partial M. Let F be a differentiable vector field on M. Then \int\_{m} div F dV = \int\_{\partial M} \langle F, n \rangle dA.

This equation is also written in terms of three differentiable functions \alpha, \beta, \gamma : M \to \mathbf{R} :

\int \int\_{M} \int (\frac{\partial \alpha }{\partial x} + \frac{\partial \beta}{\partial y} +\frac{\partial \gamma}{\partial z}) dV = \int\_{\partial M} \int (n^{1} \alpha +n^{2} \beta + n^{3} \gamma) dS.

(Stokes’ Theorem) (Thm 5.9) Let M \subset \mathbf{R}^{3} be a compact oriented two-dimensional manifold-with-boundary and n the unit outward normal on M determined by the orientation of M. Let \partial M have the induced orientation. Let T be the vector field on \partial M with ds(T) = 1 and let F be a differentiable vector field in an open set containing M. Then \int\_{M} \langle ( \nabla \times F), n \rangle dA = \int\_{\partial M} \langle F, T \rangle ds.

This equation is sometimes written

\int\_{\partial M} \alpha dx + \beta dy + \gamma dz = \int\_{M} \int [n^{1} (\frac{\partial \gamma }{\partial y } - \frac{\partial \beta }{\partial z}) + n^{2}(\frac{\partial \alpha }{\partial z} - \frac{\partial \gamma }{\partial x }) + n^{3}(\frac{\partial \beta }{\partial x } - \frac{ \partial \alpha }{\partial y}) ] dS.

(Def) If F(x) is the velocity vector of a fluid at x (at some time) then \int\_{\partial M} \langle F,n \rangle dA is the amount of fluid ‘diverging’ from M. Consequently the condition div F = 0 expresses the fact that the fluid is incompressible. If M is a disc, the \int\_{\partial M} \langle F, T \rangle ds mesures the amount that the fluid curls around the center of the disc. If this is zero for all discs, then \nabla \times F = 0 , and the fluid is called irroational.